RF and Microwave Physics

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Topics

1. Mathematical Tools
2. Electromagnetic (Maxwell's Equations, Boundary Conditions,Time varying Fields, Wave propagation).
3. Plane Waves, Electromagnetic Energy and Poynting's Theorem.
4. Transmission Lines and Waveguides.
5. Microwave Network Analysis (Impedance and Equivalent Voltages and Currents, Scattering Matrix,Signal Flow, Waveguide Excitation).

Topics

6. Microwave Resonators,CKT Design, RF Cavities.
7. Power Sources,Power Dividers and Directional Couplers.
8. RF systems for Accelerators, Linear Structures, Storage Ring Cavities.
9. Beam-Cavity Interaction, Beamloading, HOMs and Mode Damping , Wakefields, Longitudinal Effects.
10. Special Topic and Review.

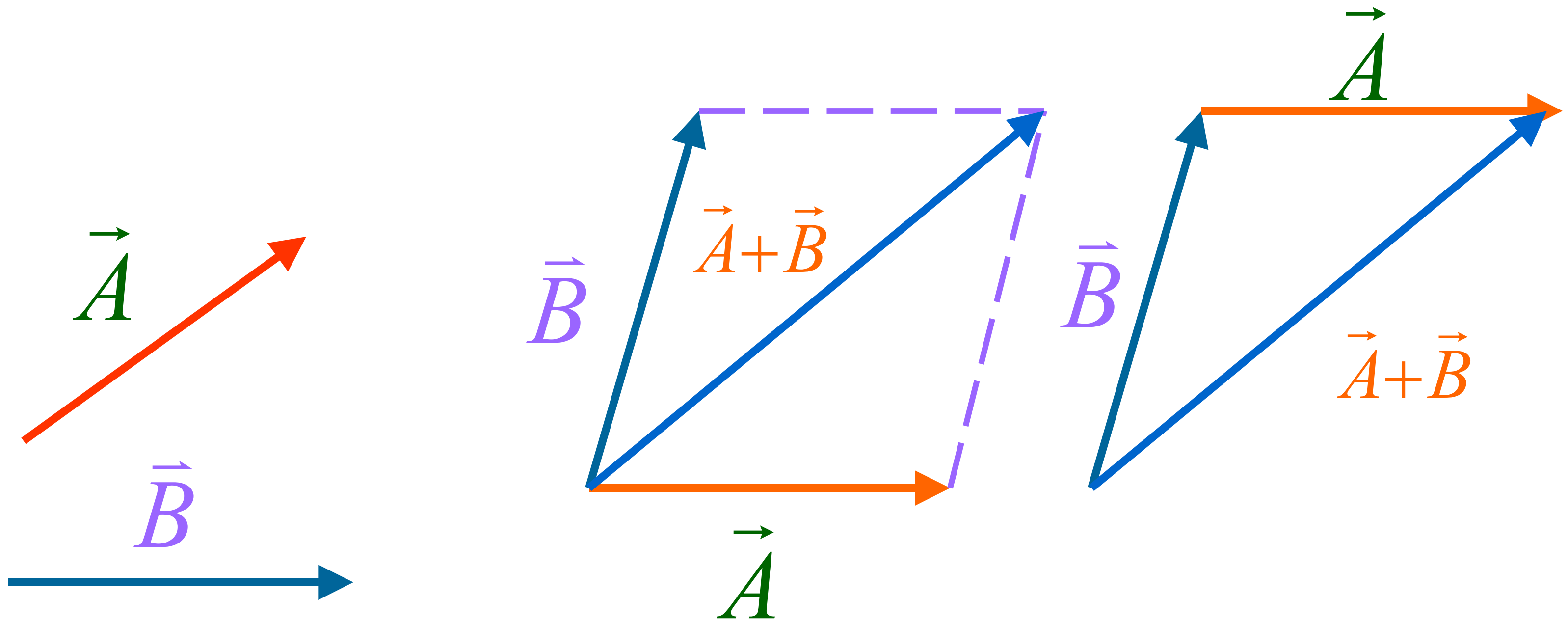
Mathematical Tools

1. Vector Analysis
2. Calculus
3. Matrices
4. Complex Numbers
5. DE/PDE
6. Fourier Series
7. Bessel and Green's Functions

Vectors

A (Euclidean) vector is an object which

1. Is added to other vector using the "Parallelogram rule"
2. Has a magnitude and direction

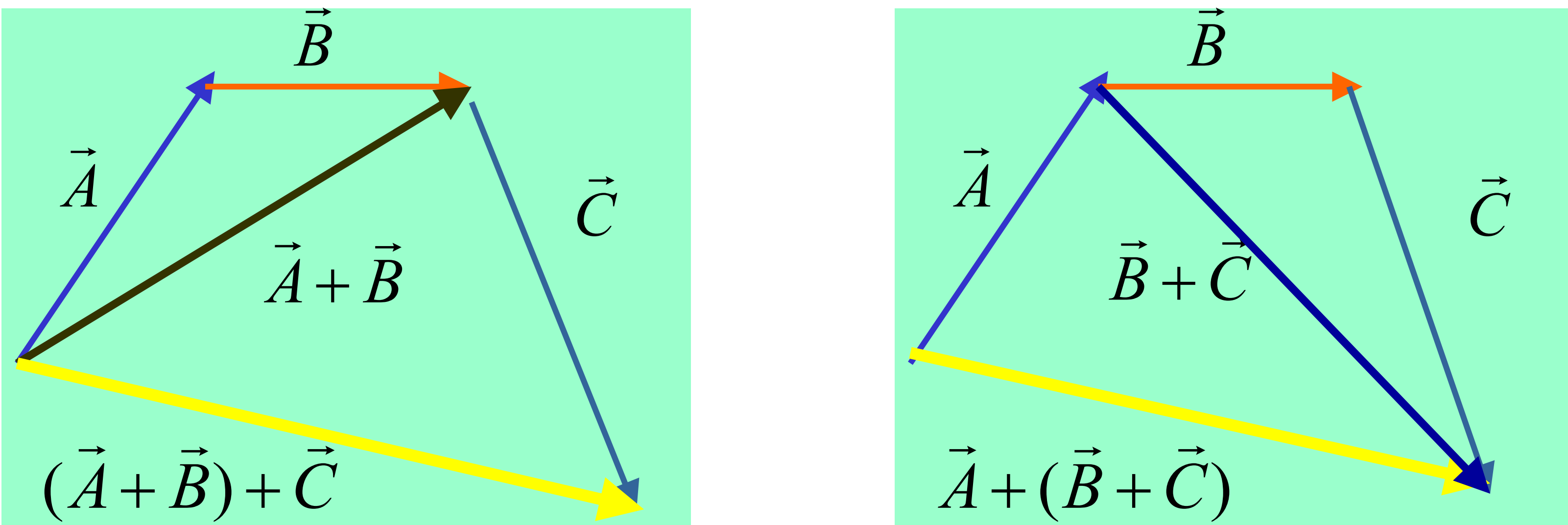


Vectors

Algebraically, $\vec{A}+\vec{B}=\vec{B}+\vec{A}$

Vector addition is also associative, I.e., when adding three (or more) vectors together we can "add the vector-sum of the first two to the third" or "add the first to the vector-sum of the last two."

Algebraically, $(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$



Vectors

Vectors can also be *scaled by multiplying them with numbers (called scalars. This is referred to as scalar-multiplication.*

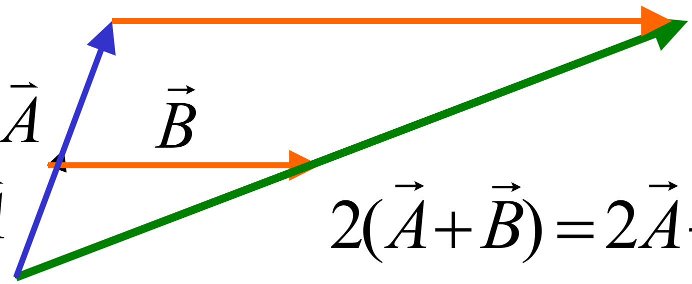
Examples: $\vec{A}$ $\quad -\vec{A}$ $\quad 2\vec{A}$

$$(\kappa + \iota)\vec{A} = \kappa\vec{A} + \iota\vec{A}$$

$$\kappa(\vec{A} + \vec{B}) = \kappa\vec{A} + \kappa\vec{B}$$

Scalar-multiplication is distributive:

$2\vec{B}$



$2\vec{A}$ $\quad \vec{B}$

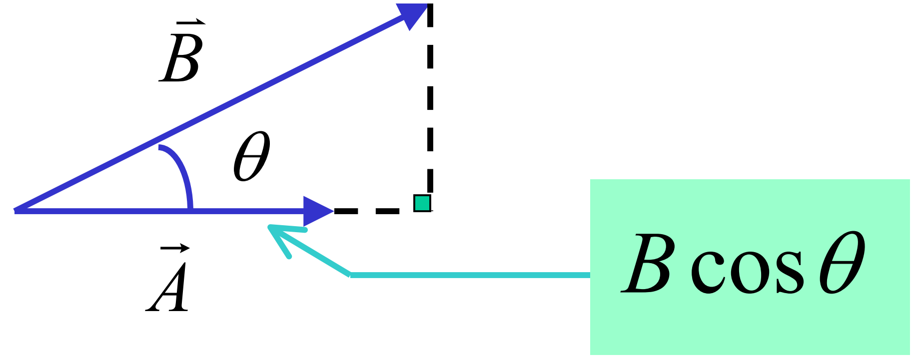
$\vec{A}$

$$2(\vec{A}+\vec{B}) = 2\vec{A}+2\vec{B}$$

Vectors

Dot-Product: Given two vectors $\vec{A}$ and $\vec{B}$, their dot-product is a multiplication rule which returns a scalar quantity. The rule is

$$\vec{A} \cdot \vec{B} = \left\| \vec{A} \right\| \left\| \vec{B} \right\| \cos\theta$$



$B \cos\theta$ is the projection of $\vec{B}$ along the direction of $\vec{A}$.

Useful Facts ...

- $\vec{A} \cdot \vec{A} = AA \cos 0 = A^2$, $\left\| \vec{A} \right\| = \sqrt{\vec{A} \cdot \vec{A}}$
- If $\vec{A} \cdot \vec{B} = AB \ (\cos\theta = 1)$, then $\vec{A}$ and $\vec{B}$ are parallel.
- If $\vec{A} \cdot \vec{B} = -AB \ (\cos\theta = -1)$, then $\vec{A}$ and $\vec{B}$ are anti-parallel.
- If $\vec{A} \cdot \vec{B} = 0 \ (\cos\theta = 0)$, then $\vec{A}$ and $\vec{B}$ are orthogonal.

Magnitude of vector-sum $\vec{A} + \vec{B}$

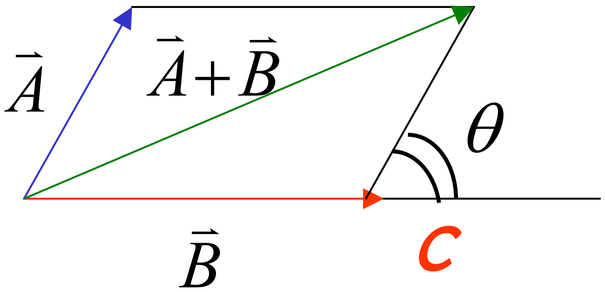
$$\left\| \vec{A} + \vec{B} \right\|^2 = (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B})$$

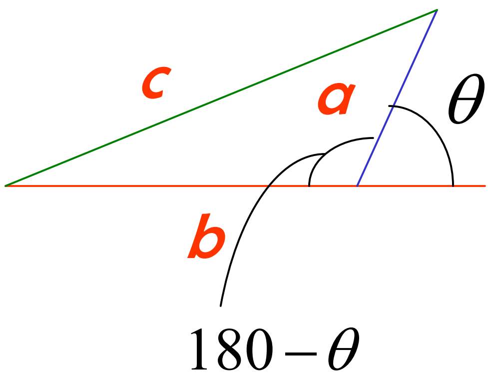
$$= (\vec{A} \cdot \vec{A}) + (\vec{A} \cdot \vec{B}) + (\vec{B} \cdot \vec{A}) + (\vec{B} \cdot \vec{B})$$

$$= \left\| \vec{A} \right\|^2 + 2(\vec{A} \cdot \vec{B}) + \left\| \vec{B} \right\|^2$$

Useful Facts ...

This is essentially the Law of Cosines $c^2 = a^2 + b^2 - 2ab\cos C$





$$\cos C = \cos(180-\theta) = -\cos\theta$$

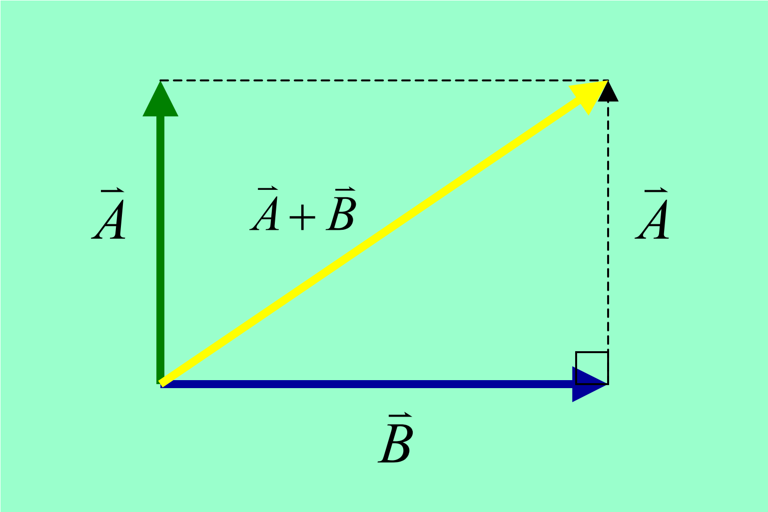
$$\text{so, } 2(\vec{A}\cdot\vec{B}) = 2ab\cos\theta = -2ab\cos C$$

Useful Facts ...

If $\vec{A}\cdot\vec{B}=0, i.e., if\ \cos\theta=0\ (\text{vectors are }\perp)$, then

$$\left\|\vec{A}+\vec{B}\right\|^2=\left\|\vec{A}\right\|^2+\left\|\vec{B}\right\|^2$$

This is the Pythagorean Theorem where $\vec{A}\ \text{and}\ \vec{B}$ are the legs of a right-triangle.



Useful Facts ...

If $\vec{A} \cdot \vec{B} = AB,$ (vectors are parallel)

$$\left\|\vec{A} + \vec{B}\right\|^2 = \left\|\vec{A}\right\|^2 + 2(AB) + \left\|\vec{B}\right\|^2$$

$$= A^2 + 2(AB) + B^2$$

$$= (A + B)(A + B)$$

$$= (\left\|\vec{A}\right\|^2 + \left\|\vec{B}\right\|^2)$$

$$\Rightarrow \qquad \left\|\vec{A} + \vec{B}\right\| = \left\|\vec{A}\right\| + \left\|\vec{B}\right\|$$

Vector operations

1. Addition (components)

Lets define two vectors, $\vec{A}$ and $\vec{B}$ as follow:

$$\vec{A} = a_1\hat{u}_1 + a_2\hat{u}_2 \quad (\hat{u}_1, \hat{u}_2)$$

, are unit vectors,

$$\vec{B} = b_1\hat{u}_1 + b_2\hat{u}_2$$

$$\hat{u}_1 = \frac{\vec{a}_1}{\|\vec{a}_1\|}, \hat{u}_2 = \frac{\vec{a}_2}{\|\vec{a}_2\|}$$

Vector operations

Define the vector sum:

$$\vec{C} = \vec{A} + \vec{B}$$

$$\vec{C} = c_1 \hat{u}_1 + c_2 \hat{u}_2$$

$$= (a_1 \hat{u}_1 + a_2 \hat{u}_2) + (b_1 \hat{u}_1 + b_2 \hat{u}_2)$$

$$= (a_1 + b_1)\hat{u}_1 + (a_2 + b_2)\hat{u}_2$$

$$c_1 = a_1 + b_1, \;\; c_2 = a_2 + b_2$$

$$\vec{C} \cdot \vec{C} = (c_1 \hat{u}_1 + c_2 \hat{u}_2) \cdot (c_1 \hat{u}_1 + c_2 \hat{u}_2)$$

Vector operations

$$\vec{C}\cdot\vec{C} = c_1 c_1(\hat{u}_1\cdot\hat{u}_1) + c_1 c_2(\hat{u}_1\cdot\hat{u}_2) + c_2 c_1(\hat{u}_2\cdot\hat{u}_1) + c_2 c_2(\hat{u}_2\cdot\hat{u}_2)$$

$$\hat{u}_1\cdot\hat{u}_1 = \hat{u}_1\cdot\hat{u}_1 = 1,\ \hat{u}_1\cdot\hat{u}_2 = \hat{u}_2\cdot\hat{u}_1 = 0$$

$$\|C\|^2 = \|c_1\|^2\|\hat{u}_1\cdot\hat{u}_1\| + \|c_2\|^2\|\hat{u}_2\cdot\hat{u}_2\|$$

$$|c| = \sqrt{|c_1|^2 + |c_2|^2}$$

Vector operations

Dot Product:

Consider two vectors $\vec{A}$ and $\vec{B}$ **:** $\vec{A}\cdot\vec{B} = AB\cos\theta$

In terms of components; $\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y$

To see this use:

$$\cos(\theta_A - \theta_B) = \cos\theta_A\cos\theta_B + \sin\theta_A\sin\theta_B$$

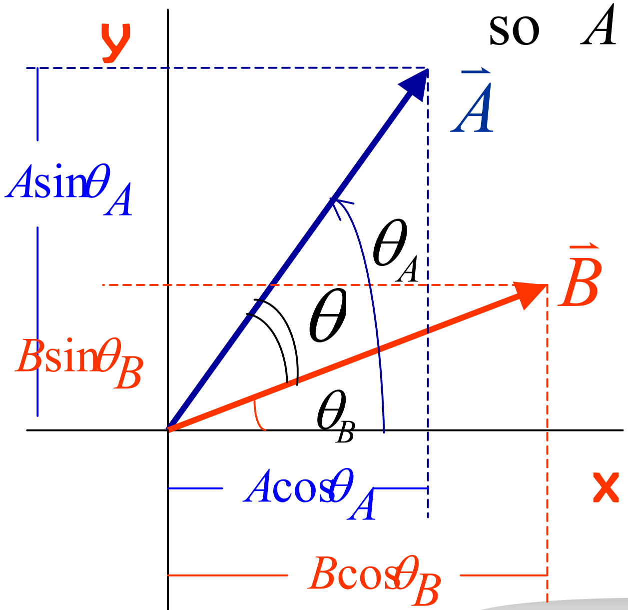
$$\text{so}\quad \vec{A}\cdot\vec{B} = AB\cos\theta \qquad (\text{note}: \theta_A = \theta_B + \theta)$$

$$= AB\cos(\theta_A - \theta_B)$$

$$= AB(\cos\theta_A\cos\theta_B + \sin\theta_A\sin\theta_B)$$

$$= (A\cos\theta_A)(B\cos\theta_B) + (A\sin\theta_A)(B\sin\theta_B)$$

$$= A_x B_x + A_y B_y$$



Vectors

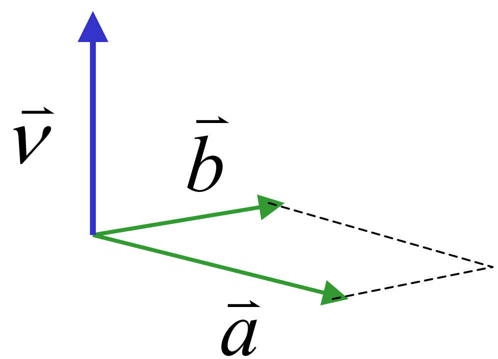
$\vec{v} = \vec{a} \times \vec{b}$ *with magnitude* $|v| = |a||b| \sin \gamma$

$\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$ *and angle* γ *betwen* $\vec{a}$ *and* $\vec{b}$.

Direction of $\vec{v} = \vec{a} \times \vec{b}$ *is* $\perp$ *to both* $\vec{a}$ *and* $\vec{b}$.

$\vec{v} = \vec{a} \times \vec{b} = [v_1, v_2, v_3]$

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}; \quad \ni v_1 = a_2 b_3 - a_3 b_2, v_2 = -(a_1 b_3 - a_3 b_1), v_3 = a_1 b_2 - a_2 b_1$$



Vector Products

- **Properties:**

$$\left(\iota\vec{a}\times\vec{b}\right)=\iota\left(\vec{a}\times\vec{b}\right)=\vec{a}\times\left(\iota\vec{b}\right),\quad \textit{(for every scalar } \iota)$$

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right),\quad \textit{(distribut ive w.r.t. addition)}$$

$$\left(\vec{a}+\vec{b}\right)\times\vec{c}=\left(\vec{a}\times\vec{c}\right)+\left(\vec{b}\times\vec{c}\right),\quad \textit{(distribut ive w.r.t addition)}$$

$$\left(\vec{b}\times\vec{a}\right)=-\left(\vec{a}\times\vec{b}\right),\quad \textit{(anticommu tative)}$$

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)\neq\left(\vec{a}\times\vec{b}\right)\times\vec{c},\quad \textit{(not associativ e, in general)}$$

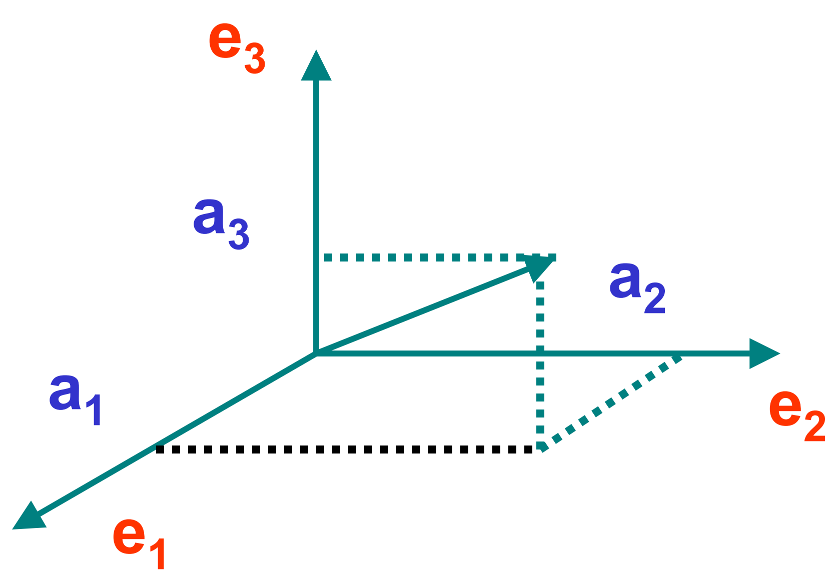
Vectors

Cartesian components of vectors

Let {e_1, e_2, e_3} be three mutually perpendicular unit vectors which form a right handed triad. Then {e_1, e_2, e_3} are said to form an *orthonormal basis*. The vectors satisfy:

$$|e_1| = |e_2| = |e_3| = 1$$

$$e_1 \times e_2 = e_3, e_1 \times e_3 = -e_2, e_2 \times e_3 = e_1$$



Vectors

We may express any vector a as a suitable combination of the unit vectors $\{e_{1,} e_{2\,,} e_3\}$. For example, we may write

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3 = \sum_{i=1}^{3} a_i e_i$$

where $\{a_1, a_2, a_3\}$ are scalars, called the components of ***a*** in the basis $\{e_{1,} e_{2\,,} e_3\}$. The components of a have ***a*** simple physical interpretation. For example, if we calculate the dot product ***a.*** e_1, we find that

$$a \cdot e_1 = \left(a_1 e_1 + a_2 e_2 + a_3 e_3\right) \cdot e_1 = a_1$$

Recall that

$$a \cdot e_1 = |a||e_1| \cos\theta(a \cdot e_1)$$

$$a_1 = a \cdot e_1 = |a| \cos\theta(a \cdot e_1)$$

Vectors

Thus, a_1 represent the projected length of the vector ***a*** in the direction of e_1. This similarly applies to a_2, a_3 .

Change of basis

Let ***a*** be a vector and let $\{e_1, e_2, e_3\}$ be a Cartesian basis. Suppose that the components of ***a*** in the basis $\{e_1, e_2, e_3\}$ are known to be $\{a_1, a_2, a_3\}$

Now, suppose that we wish to compute the components of ***a*** in a second Cartesian basis, $\{r_1, r_2, r_3\}$. This means we wish to find components $\{\alpha_1, \alpha_2, \alpha_3\}$, such that $a = \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3$ to do so, note that

$$\alpha_1 = a \cdot r_1 = \alpha_1 e_1 \cdot r_1 + \alpha_2 e_2 \cdot r_1 + \alpha_3 e_3 \cdot r_1$$

$$\alpha_2 = a \cdot r_2 = \alpha_1 e_1 \cdot r_2 + \alpha_2 e_2 \cdot r_2 + \alpha_3 e_3 \cdot r_2$$

$$\alpha_3 = a \cdot r_3 = \alpha_1 e_1 \cdot r_3 + \alpha_2 e_2 \cdot r_3 + \alpha_3 e_3 \cdot r_3$$

Vectors

This transformation is conveniently written as a matrix operation

$$\alpha = [Q][a]$$

where $[\alpha]$ is a matrix consisting of the components of ***a*** in the basis $\{r_1, r_2, r_3\}$, $[a]$ is a matrix consisting of the components of ***a*** in the basis $\{a_1, a_2, a_3\}$, and $[Q]$ is a "rotation matrix" as follows

$$[\alpha] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \quad [a] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad [Q] = \begin{bmatrix} r_1 \cdot e_1 & r_1 \cdot e_2 & r_1 \cdot e_3 \\ r_2 \cdot e_1 & r_2 \cdot e_2 & r_2 \cdot e_3 \\ r_3 \cdot e_1 & r_3 \cdot e_2 & r_3 \cdot e_3 \end{bmatrix}$$

Using index notation $\alpha_i = Q_{ij} a_j, \; Q_{ij} = r_i \cdot e_j$

Time Derivatives of Vectors

Let ***a(t)*** be a vector whose magnitude and direction vary with time, t. Suppose that {I,j,k} is a fixed basis. We may express a(t) in terms of components (a_x ,a_y ,a_z) in the basis {I,j,k} as ***a(t)*****=** a_x **i+** a_y **j+** a_z **k**. The time derivative of ***a*** is

$$\dot{a}(t) = \frac{d}{dt}a(t) = \lim_{\varepsilon\to 0}\frac{a(t+\varepsilon)-a(t)}{\varepsilon}$$

$$\dot{a}(t) = \dot{a}_x i + \dot{a}_y j + \dot{a}_z k$$

$$\frac{d}{dt}\left[\alpha(t)\cdot a(t)\right] = \dot{\alpha}(t)\cdot a(t) + \alpha(t)\cdot\dot{a}(t)$$

$$\frac{d}{dt}\left[\alpha(t)\times\beta(t)\right] = \dot{\alpha}(t)\times\beta(t) + \alpha(t)\times\dot{\beta}(t)$$

Rotating Basis

It is often convenient to express position vectors as components in a basis which rotates with time.

Let $\{e_1, e_2, e_3\}$ be a basis which rotates with instantaneous angular velocity Ω. Then,

$$\frac{de_1}{dt} = \Omega \times e_1, \ \frac{de_2}{dt} = \Omega \times e_2, \ \frac{de_3}{dt} = \Omega \times e_4$$

Gradient of a Vector Field

Let ***v*** be a vector field in three dimensional space. The gradient of ***v*** is a tensor field denoted by grad(***v***) or ∇***v*** , and is defined so that

$$(\nabla v)\cdot\alpha = \lim_{\varepsilon\to 0}\frac{v(r+\varepsilon\alpha)-v(r)}{\varepsilon}$$

for every position **r** in space and for every vector α.

Let $\{e_1, e_2, e_3\}$ be a Cartesian basis with origin O in three dimensional space. Let $r = x_1e_1 + x_2e_2 + x_3e_3$ denote the position vector of a point in space. The gradient of ***v*** in this basis is given by

$$\nabla v = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

$$[\nabla v]_{ij} \equiv \frac{\partial v_i}{\partial x_j}$$

Divergence of a Vector Field

Let ***v*** be a vector field in three dimensional space. The divergent of ***v*** is a scalar field denoted by div(***v***) or ▽•***v*** , and is defined so that

$$div(v) = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

Formally, it is defined as trace[grad(***v***)].

$$\nabla v = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

$$\nabla \cdot v = Tr(\nabla v) = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i}$$

Curl of a Vector Field

Let ***v*** be a vector field in three dimensional space. The curl of ***v*** is a vector field denoted by **curl(*v*)** or $\nabla\times$***v*** , and it is best defined in terms of its components in a given basis.

$$r = x_1 e_1 + x_2 e_2 + x_3 e_3$$

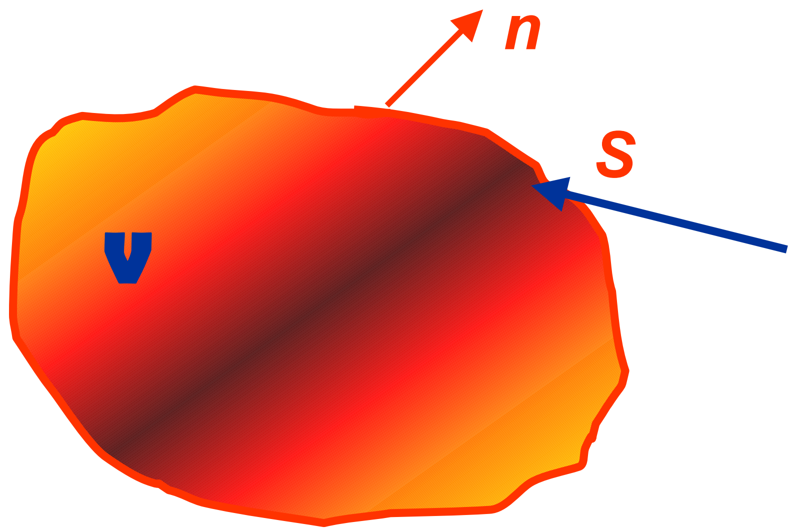
Express ***v*** as a function of the components of **r *v* = *v*(x_1,x_2,x_3).** The curl of ***v*** in this base is then given by

$$\nabla\times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) e_1 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) e_2 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) e_3$$

$$[\nabla v]i \equiv \varepsilon_{ijk} \frac{\partial v_j}{\partial x_k}$$

The Divergence Theorem

Let **V** be a closed region in three dimensional space, bounded by an oreintable surface **S.** Let n denote the unit vector normal to S, taken so that n points out of V. Let u be a vector field which is continuous and has continuous first partial derivatives in some domain containing T. Then



$$\int_V div(u)dV = \int_S u \cdot n dA$$

expressed in index notation:

$$\int_V \frac{\partial u_i}{\partial x_i} dV = \int_S u_i n_i dA$$

Definite Integral -- Properties

(i) $$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

(ii) $$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx$$

(iii) $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

(iv) $$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Integrals

$$\int f(x)g(x)dx = ?$$

Consider:

$$\left(u(x)v(x)\right)' = u'(x)v(x) + u(x)v'(x)$$

$$u(x)v(x) = \int u'(x)v(x)dx + \int u(x)v'(x)dx$$

$$\begin{cases} u = f(x) \\ dv = g(x)dx \end{cases} \Rightarrow \begin{cases} du = f'(x)dx \\ v = \int g(x)dx \end{cases}$$

Integrals

Examples

$$\int_0^1 x^2 e^x dx$$

$$\begin{cases} u = x^2 \\ dv = e^x \end{cases}$$

After integration and differentiation, we get

$$\begin{cases} du = 2xdx \\ v = e^x \end{cases}$$

$$\int_0^1 x^2 e^x dx = x^2 e^x \Big|_0^1 - \int_0^1 2xe^x dx \qquad \begin{cases} u = x \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = e^x \end{cases}$$

$$\int_0^1 xe^x dx = xe^x \Big|_0^1 - e^x \Big|_0^1$$

$$\Rightarrow \int_0^1 x^2 e^x dx = x^2 e^x \Big|_0^1 - 2xe^x \Big|_0^1 + 2e^x \Big|_0^1$$

$$\int_0^1 x^2 e^x dx = e - 2$$

Integrals

Evaluate $$\int x \tan^{-1}(x)dx$$

$$\begin{cases} u = \tan^{-1}(x) \\ dv = xdx \end{cases} \Rightarrow \begin{cases} du = \dfrac{1}{1+x^2}dx \\ v = \dfrac{1}{2}x^2 \end{cases}$$

$$\int x \tan^{-1}(x)dx = \frac{1}{2}x^2 \tan^{-1}(x) - \int \frac{1}{2}\frac{x^2}{1+x^2}dx$$

$$\int \frac{x^2}{1+x^2}dx = \int \frac{x^2+1-1}{1+x^2}dx = \int \left(1 - \frac{1}{1+x^2}\right)dx = x - \tan^{-1}(x) + C$$

$$\int x \tan^{-1}(x)dx = \frac{1}{2}x^2 \tan^{-1}(x) - \frac{x}{2} + \frac{1}{2}\tan^{-1}(x) + C$$

Integrals – trig substitution

Evaluate $\int x^3\sqrt{4-x^2}\,dx$

$$set\ x = 2\sin(t) \Rightarrow dx = 2\cos(t)dx$$

$$\int x^3\sqrt{4-x^3}\,dx = \int 8\sin^3(t)\sqrt{4-4\sin^2(t)}\,2\cos(t)dt$$

$$\int x^3\sqrt{4-x^3}\,dx = 32\int \sin^3(t)\cos^2(t)dt$$

$$\int \sin^3(t)\cos^2(t)dt = \int (1-\cos^2(t))\cos^2(t)\sin(t)dt$$

$$v = \cos(t) \Rightarrow dv = -\sin(t)dt$$

$$\int (1-\cos^2(t))\cos^2(t)\sin(t)dt = -\int (1-v^2)v^2dv = -\frac{v^3}{3}+\frac{v^5}{5}+C$$

$$\int x^3\sqrt{4-x^3}\,dx = -32\frac{v^3}{3}+32\frac{v^5}{5}+C = -\frac{4(4-x^2)^{3/2}}{3}+\frac{(4-x^2)^{5/2}}{5}+C$$

Matrices

Consider $$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

J is 3x4 matrix composed of 3 rows and 4 columns.

When the numbers of rows and columns are equal, the matrix is called a square matrix. A square matrix of order n is an (nxn) matrix.

Matrices operation

Vector, $p = \begin{bmatrix} a & b & c & d \end{bmatrix}$ is a 1x4 row matrix.

Vector, $q = \begin{bmatrix} k \\ l \\ m \\ n \end{bmatrix}$ is a4x1 column matrix.

Matrices operation

1. Addition

$$consider\ P = \begin{bmatrix} \alpha & \beta \\ \mu & \gamma \end{bmatrix} \text{ and } Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$$\text{then } T = P + Q \text{ is a } 2\times 2 \text{ matrix with :}$$

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \text{with } t_{11} = \alpha + a,\ t_{12} = \beta + b$$

$$t_{21} = \mu + c,\ t_{22} = \gamma + d$$

$$T = \begin{bmatrix} \alpha + a & \beta + b \\ \mu + c & \gamma + d \end{bmatrix}$$

Matrices operation

If λ is a constant then,

$$\lambda\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$$

$$\mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \textit{is}\, 2\times 2\, identity\;\; matrix$$

Matrices operation

$$\underbrace{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}}_{2\times 3}\underbrace{\begin{pmatrix} \alpha \\ \beta \\ \nu \end{pmatrix}}_{3\times 1}=\underbrace{\begin{pmatrix} a\alpha + b\beta + c\nu \\ d\alpha + e\beta + f\nu \end{pmatrix}}_{2\times 1}$$

$$\cancel{\underbrace{\begin{pmatrix} \alpha \\ \beta \\ \nu \end{pmatrix}}_{3\times 1}\underbrace{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}}_{2\times 3}}$$

Matrices operation

An n x n matrix A is called invertible iif there exists an n x n matrix B such that

$$AB = BA = I_n$$

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 3/2 \\ 1 & -1 \end{pmatrix}$$

$$AB = BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$$notation \quad AA^{-1} = A^{-1}A = I_n \quad (A \text{ is a } n \times n \text{ matrix})$$

$$\left(\mathrm{A}^{-1}\right)^{-1} = A \qquad (AB)^{-1} = B^{-1}A^{-1}$$

Matrices operation

Let A be a n x m matrix defined by α_{ij}, then the transpose of A, denoted A^T is the m x n matrix defined by δ_{ij} where $\delta_{ij} = \alpha_{ji}$.

1. $(X+Y)^T=X^T+Y^T$
2. $(XY)^T=Y^TX^T$
3. $(X^T)^T=X$

Matrices operation

Consider a square matrix A and define the sequence of matrices

$$A_n = I_n + \frac{1}{1!}A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots + \frac{1}{n!}A^n$$

as $n \to \infty$,

$$e^A = I_n + \frac{1}{1!}A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots + \frac{1}{n!}A^n + \ldots$$

one can write this in series notation as

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

Matrices operation

Determinants

Consider the matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. A is invertible if and only if $ad-bc\neq 0$

This number is called the determinant of A.

Determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc.$

Properties:

$$\det A = \det A^T, \quad \begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ b & d \end{vmatrix} = ad \ , \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

$$\begin{vmatrix} \lambda a & \lambda b \\ c & d \end{vmatrix} = \lambda\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ \lambda c & \lambda d \end{vmatrix}, \quad \det(AB) = \det(A)\det(B)$$

Matrices operation

In general,

$$\det(A) = \sum_{j=1}^{j=n} a_{ij} A_{ij}$$ for any fixed i

$$\det(A) = \sum_{i=1}^{i=n} a_{ij} A_{ij}$$ for any fixed j

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Eigenvalues and Eigenvectors

Let A be a square matrix. A non-zero vector C is called an eigenvector of A iff ∃ a number (real or complex) $\lambda \ni AC = \lambda C$

If λ exists, it is called an eigenvalue of A.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

$$AC_1 = 0C_1,\ AC_2 = -4C_2, \text{ and } AC_3 = 3C_3$$

$$where \quad C_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}, C_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, and \quad C_3 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$

Computing eigenvalues

$$AC = \lambda C$$

$$AI_n C = \lambda I_n C \Rightarrow AI_n C - \lambda I_n C = 0$$

$$(AI_n - \lambda I_n)C = 0 \Rightarrow \boxed{(A - \lambda I_n)C = 0}$$

This is a linear system for which the matrix coefficient is $A - \lambda I_n$.
This system has one solution if and only if the matrix coefficient is invertible,I.e. $\det(A - \lambda I_n) \neq 0$

Since the zero-vector is a solution and C is not the zero vector, we must have

$$\boxed{\det(A - \lambda I_n) = 0}$$

Computing eigenvalues

Consider matrix A:

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 0 \end{pmatrix}. \ \det\left(A - \lambda I_n\right) = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 \\ -2 & 0-\lambda \end{vmatrix} = (1-\lambda)(0-\lambda) - 4 = 0$$

which is equivalent to the quadratic equation

$$\lambda^2 - \lambda - 4 = 0$$

$$solutions:\ \lambda = \frac{1+\sqrt{17}}{2},\ and\ \lambda = \frac{1-\sqrt{17}}{2}$$

Computing eigenvalues

$$\det\left(A-\lambda I_n\right)=\det\left(A-\lambda I_n\right)^T=\det\left(A^T-\lambda I_n\right).$$

$$\textit{for any square matrix of order } 2,\ A=\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the characteristic polynomial is given by

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}=(a-\lambda)(d-\lambda)-bc=0$$

$$\Rightarrow \lambda^2-(a+b)\lambda+ad-bc=0.$$

The number (a + b) is called the **trace** of A **(denoted tr(A))** ,
and (ad - bc) is the **Determinant of A.** $\lambda^2-tr(A)\lambda+\det(A)=0.$

Complex Variables

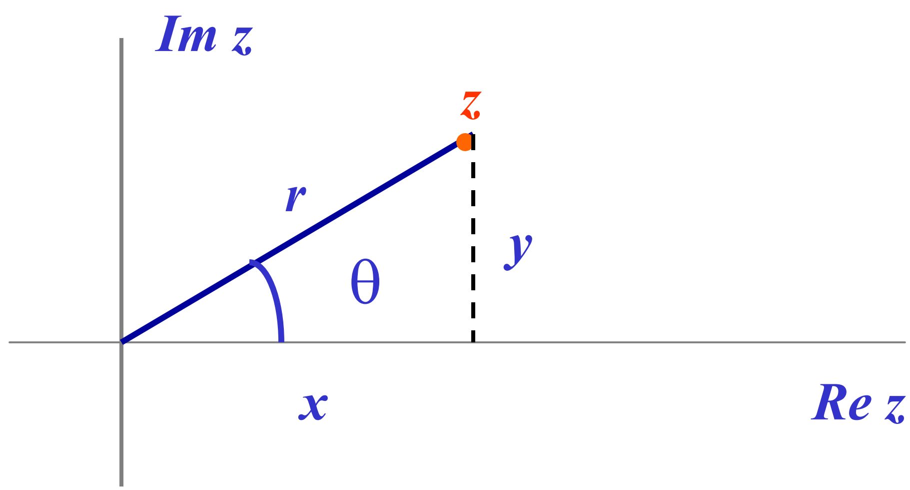
Standard notation: $$z = x + iy = re^{i\theta}$$

where $$x, y, r, \text{ and } \theta \text{ are real}, i^2 = -1$$

$$\text{and } e^{i\theta} = \cos\theta + i\sin\theta$$

x and *y* are the *real (Re z)* and *imaginary (Im z) part of z,* respectively.

$r = |z|$ is the *magnitude*, and , θ is the phase or *argument arg z*.



Complex Variables

The ***complex conjugate*** of *z* is denoted by z^*; $z^*=x-iy$.

A function $W(z)$ of the complex variable z is itself a complex number whose real and imaginary parts *U* and *V* depend on the position of z in the xy-plane. $W(z) = U(x,y) + iV(x,y)$.

$$W(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$

$$U = x^2 - y^2 \qquad V = 2xy$$

$$or \quad W = z^2 = r^2 e^{2i\theta}$$

Complex Functions

1. Exponential

$$\exp(z) = e^{z} \quad \textit{with} \quad z = x + iy$$

$$\exp(z) = e^{x}(\cos y + i\sin y)$$

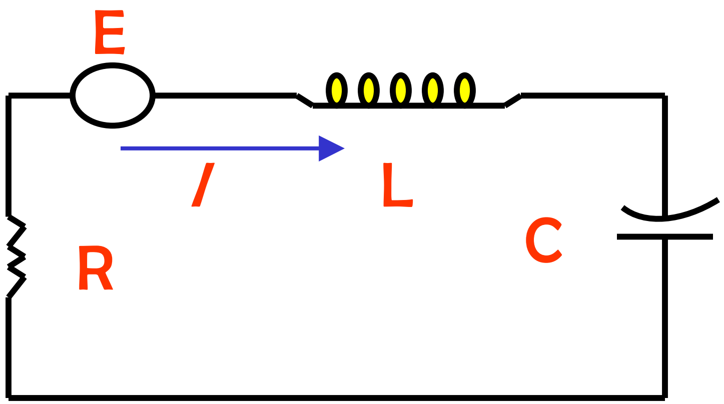
$$\frac{d}{dz}\exp(z) = \exp(z)$$

$$\textit{if} \;\; z = x + iy \;\; \textit{and} \;\; w = u + iv, \;\; \textit{then}$$

$$\begin{aligned} \textit{exp}(z + w) &= e^{x+u}\left[\cos(y + v) + i\sin(y + y)\right] \\ &= e^{x}e^{u}\left[\cos y\cos v - \sin y\sin v + i(\sin y\cos v + \cos y\sin v)\right] \\ &= e^{x}e^{u}(\cos y + i\sin y)(\cos v + i\sin v) \\ &= \exp(z)\exp(w) \end{aligned}$$

Complex Functions

Circuit problem



$$V_R = RI$$
$$V_L = L\frac{di}{dt}$$
$$i_C = C\frac{dV}{dt}$$

$$V(t) = A\sin(\omega t + \phi) \Rightarrow V = \mathrm{Im}(Ae^{i\phi}e^{i\omega t}) = \mathrm{Im}(Be^{i\omega t})$$

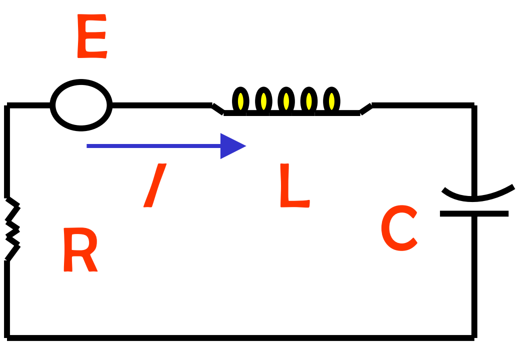
$$I = \mathrm{Im}(Ce^{i\omega t})$$

$$\frac{d}{dt}Ae^{i\omega t} = i\omega Ae^{i\omega t}. \quad if \quad I = be^{i\omega t},$$

$$\Rightarrow V = i\omega LI \ \ (for\ inductor)\ and\ i\omega VC = I,\ or\ V = \frac{I}{i\omega C} \quad for\ a\ capacitor.$$

Complex Functions

Kirchoff's law:



$$i\omega LI + \frac{I}{i\omega C} + RI = ae^{i\omega t} \qquad \left(E = a^{i\omega t}\right)$$

$$i\omega Lb + \frac{b}{i\omega C} + Rb = a$$

$$\Rightarrow \quad b = \frac{a}{R + i\left(\omega L - \frac{1}{\omega C}\right)}$$

$$b = \frac{a}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{i\phi}, \qquad \tan\phi = \frac{\omega L - \frac{1}{\omega C}}{R}$$

$$I = \mathrm{Im}(be^{i\omega t}) = \mathrm{Im}\left(\frac{a}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{i(\omega t + \phi)}\right)$$

$$= \frac{a}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin(\omega t + \phi)$$

Differential Equations

1st order DE has the following form:

$$\frac{dy}{dx} + P(x)y = q(x)$$

The general solution is given by

$$y = \frac{\int u(x)q(x) + C}{u(x)},$$

$$u(x) = \exp\left(\int p(x)dx\right)$$

U(x) is called the integrating factor.

Differential Equations

Example 1

Find the particular solution of $y' + \tan(x)y = \cos^2(x),\ \ y(0) = 2.$

- step 1: identify *p(x)* and *q(x).*

$$p(x) = \tan(x) \quad and\ q(x) = \cos^2(x)$$

- step 2: Evaluate the integrating factor

$$u(x) = e^{\int \tan(x)\,dx} = e^{-\ln(\cos(x))} = e^{\ln(\sec(x))} = \sec(x)$$

- We have

$$\int \sec(x)\cos^2(x)dx = \int \cos(x)dx = \sin(x)$$

$$y = \frac{\sin(x) + C}{\sec(x)} = (\sin(x) + C)\cos(x),\ \ y(0) = C = 2$$

$$y = (\sin(x) + 2)\cos(x)$$

Differential Equations

Example 2

Find solution to

$$\cos^2(t)\sin(t)y' = -\cos^3(t)y + 1, \quad y(\pi/4) = 0.$$

Rewrite the equation:

$$y' = -\frac{\cos^3(t)}{\cos^2(t)\sin(t)}y + \frac{1}{\cos^2(t)\sin(t)} = -\frac{\cos(t)}{\sin(t)}y + \frac{1}{\cos^2(t)\sin(t)}$$

$$\rightarrow y' + \frac{\cos(t)}{\sin(t)}y = \frac{1}{\cos^2(t)\sin(t)}$$

Hence the integration factor is given by

$$\boxed{u(t) = e^{-\int \frac{\cos(t)}{\sin(t)}dt} = e^{\ln|\sin(t)|} = \sin(t)}$$

Differential Equations

Example 2

The general solution can be obtained as

$$y = \frac{\int \sin(t) \frac{1}{\cos^2(t)\sin(t)} dt + C}{\sin(t)}$$

Since we have

$$\int \sin(t) \frac{1}{\cos^2(t)\sin(t)} dt = \int \frac{1}{\cos^2(t)} dt = \tan(t),$$

Differential Equations

Example 2

We get

$$y = \frac{\tan(t) + C}{\sin(t)} = \frac{1}{\cos(t)} + \frac{C}{\sin(t)} = \sec(t) + C\csc(t)$$

The initial condition $y(\frac{\pi}{4}) = 0$ implies

$$\sqrt{2} + C\sqrt{2} = 0, \Rightarrow C = -1$$

$$\boxed{y(t) = \sec(t) - \csc(t)}$$

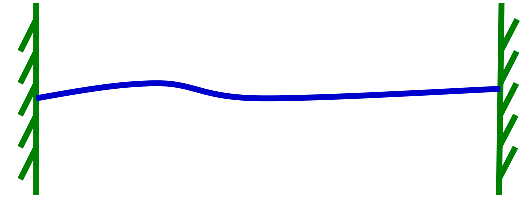
Separation of Variables-PDE

This method can be applied to partial differential equations, especially with constant coefficients in the equation. Consider one-dim wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(x,t) \textit{ is the displacement (deflection) of the stretched string.}$$

$$u(0,t) = 0 \quad u(L,t) = 0 \quad \forall t$$ (BC's)

$$u(x,0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = g(x)$$ (IC's)



Basic idea:

1. Apply the method of separation to obtain two *ordinary* DE's
2. Determine the solutions that satisfy the bc's.
3. Use Fourier series to superimpose the solutions to get final solution that satisfies both the wave equation and the initial conditions.

Separation of Variables-PDE

We seek a solution of the form

$$u(x,t) = \mathrm{X}(x)\mathrm{T}(t)$$

Differentiating, we get

$$\frac{\partial u}{\partial t} = \mathrm{X}(x)\dot{\mathrm{T}}(t) \qquad \Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial^2 u}{\partial t^2} = \mathrm{X}(x)\ddot{\mathrm{T}}(t)$$

and

$$\frac{\partial u}{\partial x} = \mathrm{X}'(x)\mathrm{T}(t) \qquad \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} = \mathrm{X}''(x)\mathrm{T}(t)$$

Thus the wave equation becomes

$$\mathrm{X}''(x)\mathrm{T}(t) = \frac{1}{c^2}\mathrm{X}(x)\ddot{\mathrm{T}}(t),$$

dividing by the product $\mathrm{X}(x)\mathrm{T}(t)$

$$\boxed{\frac{\mathrm{X}''}{\mathrm{X}} = \frac{\ddot{\mathrm{T}}}{c^2\mathrm{T}}}$$

Separation of Variables-PDE

$$\frac{X''}{X} = \frac{\ddot{T}}{c^2 T} = constant \; = c$$

$$X'' = cX$$

$$\ddot{T} = cT$$

We allow the constant to take any value and then show that only certain values are allowed to satisfy the boundary conditions. We consider the three possible cases for *c*, namely $c = p^2$ positive, $c = 0$, and $c = -p^2$. These give us three distinct types of solution that are restricted by the initial and boundary conditions.

With $c = 0$

$$X'' = 0 \qquad \Rightarrow X(x) = Ax + B$$

$$\ddot{T} = 0 \qquad \Rightarrow T(t) = Dt + E$$

Separation of Variables-PDE

$c = p^2$

$$X'' - p^2 X = 0$$

$$\ddot{T} - c^2 p^2 T = 0$$

$$X(x) = e^{\lambda x}, \qquad \Rightarrow X''(x) = \lambda^2 e^{\lambda x} = \lambda^2 X(x)$$

$$\lambda^2 X - p^2 X = 0, \qquad \Rightarrow \lambda^2 = p^2 \qquad \Rightarrow \lambda = \pm p$$

Solution:

$$X(x) = Ae^{px} + Be^{-px}$$

BC's in x$\Rightarrow$A=0, B=0. Trivial solution

Separation of Variables-PDE

$$c = -p^2$$

$$X'' + p^2 X = 0$$

$$\ddot{T} + c^2 p^2 T = 0$$

$$X(x) = e^{\lambda x}$$

where $\lambda^2 = -p^2 \quad \Rightarrow \lambda = \pm ip$

Thus the solution is $\quad X(x) = A\cos px + B\sin px$

BC at $x = 0 \Rightarrow A = 0$, *at* $x = L$ $\; X(L) = B\sin pL$

if $B = 0$, *we have the trivial solution. Non-trivial solution* $\Rightarrow \sin pL = 0$

$\Rightarrow pL = n\pi$, n *is an integer.*

Separation of Variables-PDE

Similarly;

$$\mathrm{T}(t) = D\cos pct + E\sin pct$$

$p = n\pi/L$. Thus, a solution for $u(x,t)$ is

$$u(x,t) = A\sin\frac{n\pi}{L}x\left(D\cos\frac{n\pi c}{L}t + E\sin\frac{n\pi c}{L}t\right)$$

$$u(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi}{L}x\left(D_n\cos\frac{n\pi c}{L}t + E_n\sin\frac{n\pi c}{L}t\right)$$

We can set $A=1$ without any loss of generality.

Separation of Variables-PDE

Applying IC's. Setting t=0.

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi}{L} x$$

since $\sin(0) = 0$ *and* $\cos(0) = 1,$

$$f(x) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi}{L} x$$

To determine the constants, D_n, we multiply both sides of the equation by $\sin\frac{m\pi}{L}x$ and integrate from *x=0 to x=L.*

$$\int_0^L f(x)\sin\frac{m\pi}{L}x\,dx = \int_0^L\left(\sum_{n=1}^{\infty} D_n \sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}x\right)dx.$$

$$\int_0^L f(x)\sin\frac{m\pi}{L}x\,dx = \sum_{n=1}^{\infty}\left(\int_0^L D_n \sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}x\right)dx$$

Separation of Variables-PDE

Using orthogonality condition:

$$\int_0^L f(x)\sin\frac{m\pi}{L}x\,dx = D_m\frac{L}{2}.$$

Replacing m by n:

$$D_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx\,.$$

the other IC requires the time derivative of $u(x,t)$.

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\frac{n\pi c}{L}\sin\frac{m\pi}{L}x\left(E_n\cos\frac{n\pi c}{L}t - D_n\sin\frac{n\pi c}{L}t\right).$$

at $t = 0,$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty}\frac{n\pi c}{L}E_n\sin\frac{n\pi}{L}x.$$

Separation of Variables-PDE

using IC,

$$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} E_n \sin \frac{n\pi}{L} x.$$

Repeat the same procedure

$$\int_0^L g(x) \sin \frac{m\pi}{L} x dx = \frac{m\pi c}{L} E_m \frac{L}{2}.$$

$$\Rightarrow E_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi}{L} x dx$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left(D_n \cos \frac{n\pi c}{L} t + E_n \sin \frac{n\pi c}{L} t \right).$$

Fourier series

$$A_0 + \sum_{n=1}^{\infty}(A_n \cos(nx) + B_n \sin(nx)).$$

A Fourier polynomial is an expression of the form

$$F_n(x) = a_0 + (a_1 \cos(x) + b_1 \sin(x)) + \ldots + (a_n \cos(nx) + b_n \sin(nx))$$

Which may be written as

$$F_n = a_0 + \sum_{k=1}^{n}(a_k \cos(kx) + b_k \sin(kx)).$$

The constants $a_0, a_i \text{ and } b_i, i = 1,\ldots,n,$ are called the coefficients of $F_n(x)$.

Fourier series

The Fourier polynomials are 2 π-periodic functions.

$$F_n = a_0 + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)).$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x)dx,$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F_n(x)\cos(kx)dx, 1 \le k \le n$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F_n(x)\sin(kx)dx, 1 \le k \le n$$

Fourier series

Example

Find the Fourier series of the function $f(x) = x, \; -\pi \le x \le \pi.$

Since $f(x)$ is odd, then $a_n = 0, \; for \; n \ge 0.$ For any $n \ge 1,$ we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(x)dx = \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_{-\pi}^{\pi}$$

$$\Rightarrow b_n = -\frac{2}{n}\cos(n\pi) = \frac{2}{n}(-1)^{n+1}.$$

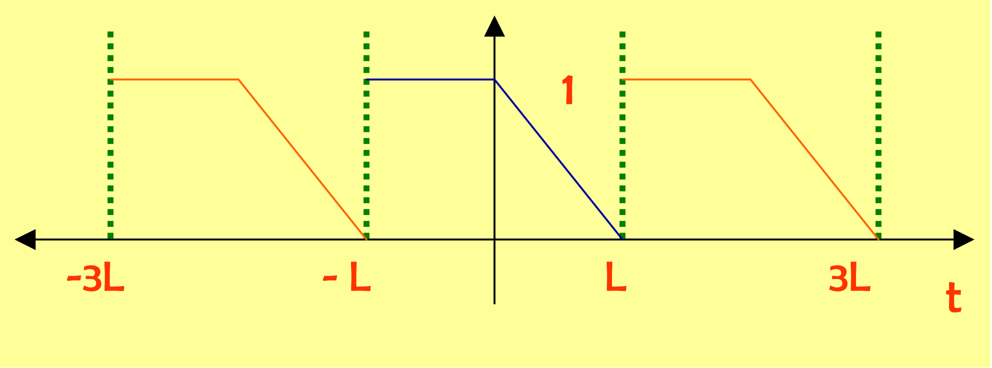
$$Hence \quad f(x) \sim 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3}\cdots\right).$$

Fourier series

Example

Find the Fourier series of the function with period 2L defined by

$$f(t)=\begin{cases}1 & -L<t<0\\ 1-\dfrac{t}{L} & 0<t<L\end{cases}$$



$$T=2L,\omega=\frac{2\pi}{T}=\frac{\pi}{L}$$

Fourier series given by

$$f(t)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\omega t)+bn\sin(n\omega t)$$

Fourier series

Coefficients found by evaluating

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)dt\,,\; b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)dt$$

Calculating

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)dt = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$= \frac{1}{L}\left\{\int_{-L}^{0}\cos\left(\frac{n\pi t}{L}\right)dt + \int_{0}^{L}\left(1-\frac{t}{L}\right)\cos\left(\frac{n\pi t}{L}\right)dt\right\}$$

$$= \frac{1}{L}\left\{\left[\frac{L}{n\pi}\sin\left(\frac{n\pi t}{L}\right)\right]_{-L}^{0} + \left[\left(1-\frac{t}{L}\right)\frac{L}{n\pi}\sin\left(\frac{n\pi t}{L}\right)\right]_{0}^{L} + \int_{0}^{L}\frac{1}{L}\frac{L}{n\pi}\sin\left(\frac{n\pi t}{L}\right)dt\right\}$$

Fourier series

$$a_n = \frac{1}{n\pi L}\int_0^L \sin\left(\frac{n\pi t}{L}\right)dt$$

$$= \frac{1}{n\pi L}\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi t}{L}\right)\right]_0^L$$

$$= \frac{1-\cos(n\pi)}{n^2\pi^2} = \frac{1-(-1)^n}{n^2\pi^2}$$

$$\therefore\ a_n = 0\ \ if\ \ n\ is\ even,\ \ a_n = \frac{2}{n^2\pi^2}\ if\ \ n\ is\ odd,$$

$$\Rightarrow a_{2m} = 0,\ \ a_{2m+1} = \frac{2}{(2m+1)^2\pi^2}$$

Fourier series

Calculate a_0

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$= \frac{1}{L}\left\{\int_{-L}^{0} 1dt + \int_{0}^{L} 1 - \frac{t}{L}dt\right\}$$

$$= \frac{1}{L}\left\{[t]_{-L}^{0} + \left[t - \frac{t^2}{2L}\right]_0^L\right\}$$

$$= \frac{1}{L}\left\{L + L - \frac{L^2}{2L}\right\} = \frac{3}{2}$$

$$\therefore a_0 = 3/2$$

Fourier series

Calculating b_n

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)dt = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

$$= \frac{1}{L}\left\{\int_{-L}^{0}\sin\left(\frac{n\pi t}{L}\right)dt + \int_{0}^{L}\left(1-\frac{t}{L}\right)\sin\left(\frac{n\pi t}{L}\right)dt\right\}$$

$$= \frac{1}{L}\left\{\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi t}{L}\right)\right]_{-L}^{0} - \left[\left(1-\frac{t}{L}\right)\frac{L}{n\pi}\cos\left(\frac{n\pi t}{L}\right)\right]_{0}^{L} - \int_{0}^{L}\frac{1}{L}\frac{L}{n\pi}\cos\left(\frac{n\pi t}{L}\right)dt\right\}$$

$$= \frac{\cos(n\pi)-1}{n\pi} + \frac{1}{n\pi} - \frac{1}{n\pi L}\int_{0}^{L}\cos\left(\frac{n\pi t}{L}\right)dt$$

$$= \frac{(-1)^n}{n\pi} - \frac{1}{n\pi L}\left[\frac{L}{n\pi}\sin\left(\frac{n\pi t}{L}\right)\right]_{0}^{L} = \frac{(-1)^n}{n\pi} \qquad \therefore b_n = \frac{(-1)^n}{n\pi}$$

Fourier series

We now know that

$$a_{2n} = 0, \qquad a_{2n+1} = \frac{2}{(2n+1)^2 \pi^2} \qquad n = 1,2,3,\ldots$$

$$a_0 = \frac{3}{2}$$

$$b_n = \frac{(-1)^n}{n\pi} \qquad n = 1,2,3,\ldots$$

$$f(t) \sim \frac{3}{4} + \sum_{n=1}^{\infty} \frac{2}{(2n+1)^2 \pi^2} \cos\left(\frac{(2n+1)\pi t}{L}\right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \sin\left(\frac{n\pi t}{L}\right)$$

Fourier transform

The continuous time Fourier transform of $x(t)$ is defined as

$$\chi(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt,$$

and the inverse transform is defined as

$$x(t) = \int_{-\infty}^{\infty} \chi(f) e^{i2\pi ft} df$$

A common notation is to define the Fourier transform in terms of $i\omega$ as

$$X(i\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt,$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(i\omega) e^{i\omega t} d\omega$$

Fourier transform properties symmetry

$$\chi(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt$$

$$\chi(f) = \int_{-\infty}^{\infty} (x_e(t) + x_o(t))(\cos(2\pi ft) - i\sin(2\pi ft))dt.$$

The odd components of the integrand contribute zero to the integral.
Hence

$$\chi(f) = \int_{-\infty}^{\infty} x_e(t)(\cos(2\pi ft) + i\int_{-\infty}^{\infty} - x_o(t)\sin(2\pi ft)dt,$$

$$\chi(f) = \chi_r(f) + i\chi_i(f),$$

where

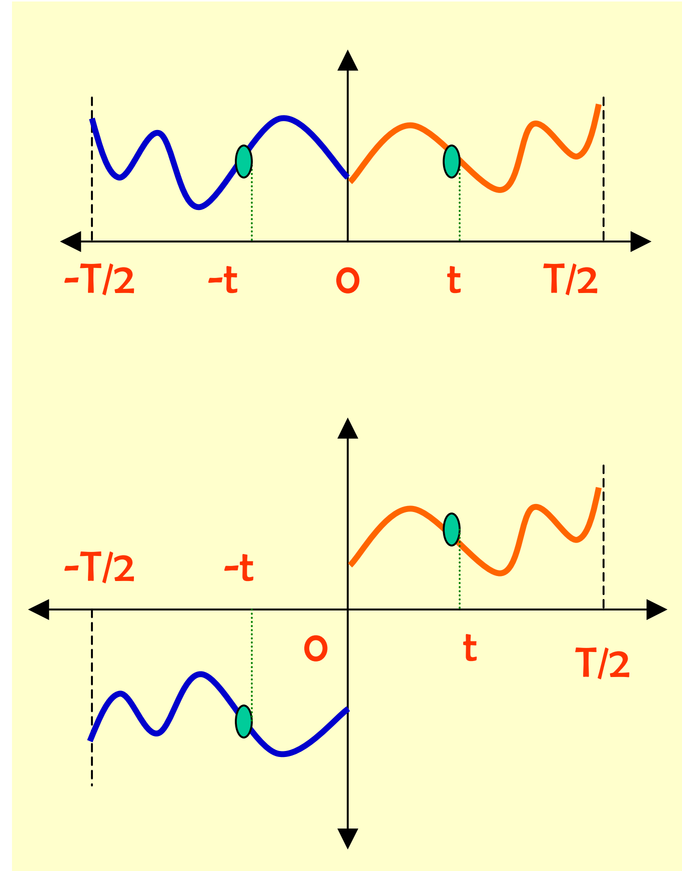
$$\chi_r(f) = \int_{-\infty}^{\infty} x_e(t)\cos(2\pi ft)dt,$$

$$\chi_i(f) = -\int_{-\infty}^{\infty} x_o(t)\sin(2\pi ft)dt.$$

Odd and Even Functions

Even	Odd
$f(-t) = f(t)$	$f(-t) = -f(-t)$
Symmetric	Anti-symmetric
Cosines	Sines
Transform is real* imaginary	Transform is

*for real-valued signals



- Important property of even and odd functions for any L,

$$\int_{-L}^{L} f(t)dt = 2\int_{0}^{L} f(t)dt$$ *If f is even*

$$\int_{-L}^{L} f(t)dt = 0$$ *If f is odd*

Convolution Theorem

Let *F*, *G*, *H* denote the Fourier Transforms of signals *f*, *g*, and *h* respectively.

g = f*h	***g = fh***
implies	implies
G = FH	**G = F*H**

Convolution in one domain is multiplication in the other and vice versa.

$$\Im(f(t) * g(t) = \Im(f(t))\Im(g(t))$$

$$\Im(f(t)g(t) = \Im(f(t)) * \Im(g(t))$$

Convolution

$$\Im(f(t) * g(t)) = \Im\left(\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau\, e^{-i2\pi\omega t}dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-\tau)g(\tau)e^{-i2\pi\omega t}d\tau dt$$

$$\Im(f(t) * g(t)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-\tau)g(\tau)e^{-i2\pi\omega t}d\tau dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)g(\tau)e^{-i2\pi\omega(u+\tau)}d\tau du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{-i2\pi\omega u}g(\tau)e^{-i2\pi\omega\tau}d\tau du$$

$$= \int_{-\infty}^{\infty} f(u)e^{-i2\pi\omega u}du \int_{-\infty}^{\infty} g(\tau)e^{-i2\pi\omega\tau}d\tau$$

$$\Im(f(t) * g(t)) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi\omega t}dt \int_{-\infty}^{\infty} g(t)e^{-i2\pi\omega t}dt$$

$$= \Im(f(t))\Im(g(t))$$

Convolution

$$\Im(f(t) * g(t)) = \Im(f(t))\Im(g(t))$$

$$\Im(f(t)g(t)) = \Im(f(t)) * \Im(g(t))$$